Introduction to Geophysics

[ES 3004 / ES 7020]

Semester 2, AY2025–2026

Nanyang Technological University

Tutorial
Half-Space Cooling Model of the Oceanic Lithosphere

1. Physical setting

The *half-space cooling model* is a simple representation of how the oceanic lithosphere cools as it moves away from a mid-ocean ridge. The basic assumptions are:

- At time $t = 0$ a semi-infinite half-space ($z > 0$) is at uniform mantle temperature T_m.
- The surface at $z = 0$ is suddenly held at a constant, colder temperature T_0 (seafloor temperature).
- Heat is transported only by vertical conduction; material properties are constant and the thermal diffusivity is κ.
- The domain is semi-infinite: $T(z,t) \to T_m$ as $z \to \infty$.

This represents a young plate newly formed at a mid-ocean ridge that cools and thickens conductively as it ages.

2. Governing equation and boundary conditions

Let $T(z,t)$ be the temperature as a function of depth z (positive downward) and time t. One-dimensional heat conduction gives

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2}, \qquad z > 0,\ t > 0. \tag{1}$$

Initial and boundary conditions:

$$T(z,0) = T_m, \qquad z > 0 \quad \text{(initially hot half-space)}, \tag{2}$$

$$T(0,t) = T_0, \qquad t > 0 \quad \text{(fixed cold surface)}, \tag{3}$$

$$T(z,t) \to T_m, \qquad z \to \infty \quad \text{(mantle interior)}. \tag{4}$$

It is convenient to non-dimensionalise the temperature via

$$\theta(z,t) = \frac{T(z,t) - T_0}{T_m - T_0}. \tag{5}$$

Then θ satisfies the same diffusion equation,

$$\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial z^2}, \tag{6}$$

with simpler conditions

$$\theta(z,0) = 1, \qquad z > 0, \tag{7}$$

$$\theta(0,t) = 0, \qquad t > 0, \tag{8}$$

$$\theta(z,t) \to 1, \qquad z \to \infty. \tag{9}$$

3. Similarity solution

We look for a similarity solution of the form

$$\theta(z,t) = f(\eta), \qquad \eta = \frac{z}{2\sqrt{\kappa t}}. \tag{10}$$

Using the chain rule,

$$\frac{\partial \theta}{\partial t} = f'(\eta)\,\frac{\partial \eta}{\partial t} = f'(\eta)\left(-\frac{\eta}{2t}\right), \tag{11}$$

$$\frac{\partial \theta}{\partial z} = f'(\eta)\,\frac{\partial \eta}{\partial z} = f'(\eta)\frac{1}{2\sqrt{\kappa t}}, \tag{12}$$

$$\frac{\partial^2 \theta}{\partial z^2} = f''(\eta)\left(\frac{1}{2\sqrt{\kappa t}}\right)^2 = f''(\eta)\frac{1}{4\kappa t}. \tag{13}$$

Substituting into Eq. (6) gives

$$-\frac{\eta}{2t}f'(\eta) = \kappa\frac{1}{4\kappa t}f''(\eta) \quad \Longrightarrow \quad f''(\eta) + 2\eta f'(\eta) = 0. \tag{14}$$

Define $g(\eta) = f'(\eta)$. Then

$$g'(\eta) + 2\eta g(\eta) = 0, \tag{15}$$

which can be solved by separation of variables:

$$\frac{g'}{g} = -2\eta \quad \Longrightarrow \quad \ln g = -\eta^2 + C_1 \quad \Longrightarrow \quad g(\eta) = Ce^{-\eta^2}, \tag{16}$$

where $C = e^{C_1}$ is a constant. Integrating once more,

$$f(\eta) = C\int_0^{\eta} e^{-s^2}\,ds + C_2. \tag{17}$$

The integral is related to the error function,

$$\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-s^2}\,ds, \tag{18}$$

so we can write

$$f(\eta) = A\ \mathrm{erf}(\eta) + B, \tag{19}$$

for suitable constants A and B.

4. Applying boundary conditions

We now impose the boundary conditions on θ:

- At the surface $z = 0$ we have $\eta = 0$ and $\theta(0,t) = 0$ for all t, so

$$f(0) = A\ \mathrm{erf}(0) + B = 0 \quad \Longrightarrow \quad B = 0. \tag{20}$$

- As $z \to \infty$, $\eta \to \infty$ and $\theta \to 1$, so

$$f(\infty) = A\ \mathrm{erf}(\infty) = A \cdot 1 = 1 \quad \Longrightarrow \quad A = 1. \tag{21}$$

Thus

$$\theta(z,t) = f(\eta) = \mathrm{erf}\left(\frac{z}{2\sqrt{\kappa t}}\right). \tag{22}$$

Returning to dimensional temperature,

$$T(z,t) = T_0 + (T_m - T_0)\ \mathrm{erf}\left(\frac{z}{2\sqrt{\kappa t}}\right). \tag{23}$$

It is often convenient to write $\Delta T = T_m - T_0$ and box the final result:

$$\boxed{T(z,t) = T_0 + \Delta T\ \mathrm{erf}\left(\frac{z}{2\sqrt{\kappa t}}\right)}. \tag{24}$$

5. MATLAB exercise: implementing the half-space cooling model

In this exercise you will:

1. Implement the analytical solution for $T(z, t)$ in MATLAB.
2. Plot the temperature profile $T(z, t)$ for different plate ages.
3. Examine how the thermal boundary layer thickens with time.

Suggested parameter values:

- Surface temperature: $T_0 = 0$ °C,
- Mantle temperature: $T_m = 1300$ °C,
- Thermal diffusivity: $\kappa = 10^{-6}$ m^2/s,
- Depth range: $z = 0$–150 km,
- Plate ages: $t = 1$, 10, 50, 100 Myr.

Questions for students:

1. How does the thickness of the cold thermal boundary layer scale with the square root of age in your plot?
2. How would you modify the script to compute surface heat flow $q(t) = -k\,\partial T/\partial z|_{z=0}$ and plot it as a function of age?
3. Compare your results qualitatively with observed seafloor depth and heat-flow patterns in the oceans.