Stress and Strain

Stress and Strain

Deformation or strain is the direct result of the forces applied: these may be body forces (e.g., gravitational force), which act on a volume, or surface forces (e.g., the forces applied to the edge of a plate, or a tectonic fault). Surface forces are measured as the stress, or force per unit area. Strain is measured as relative changes in the length of lines (stretches) and as changes in angle (shear strain).

How do we study stress and strain?

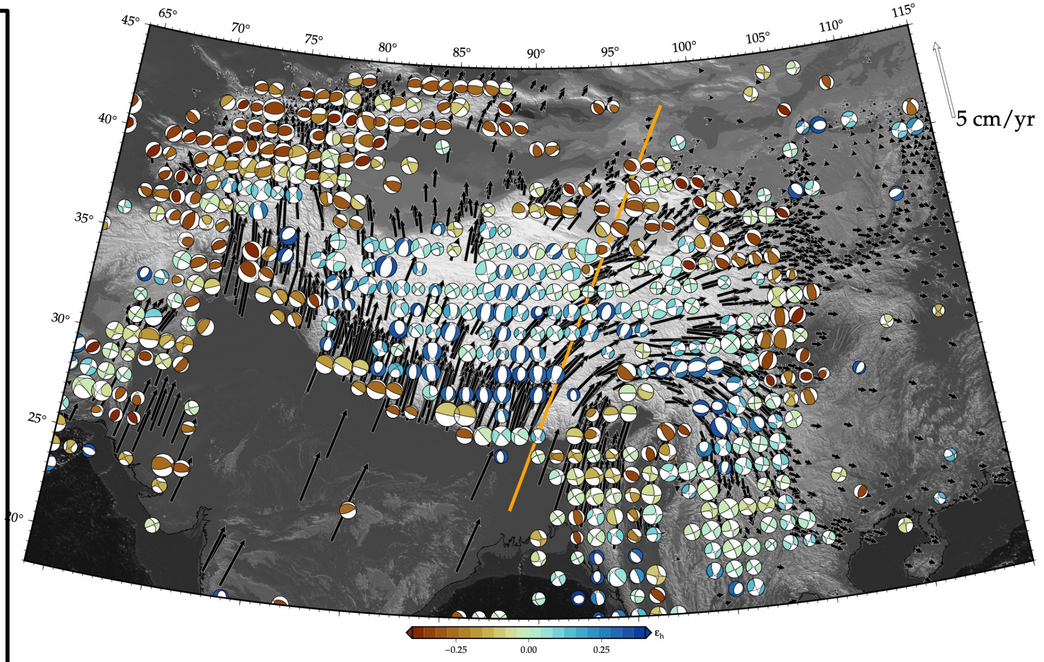
Continuum Mechanics

Continuum mechanics is a branch of physics and engineering that models materials as continuous media, ignoring their discrete particulate nature to study their deformation and motion. It uses the **principles of classical mechanics and conservation laws (mass, momentum, energy)** to develop mathematical models that describe the behavior of solids and fluids under forces.

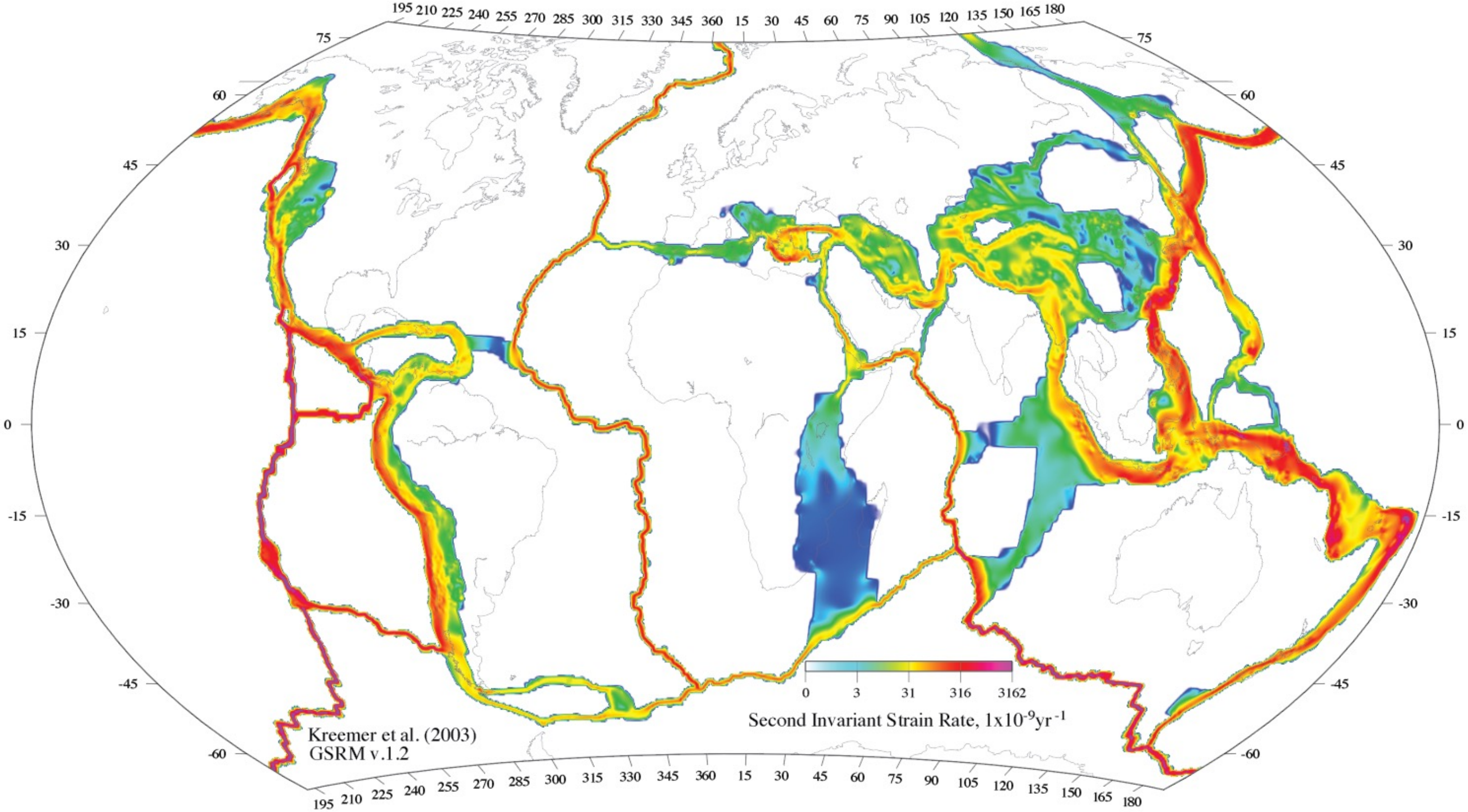
In order to understand the processes that shape our planet's surface geology and its evolution over time, and to formulate a theory that can link observations to dynamics, we need a description of both deformation and driving forces. Continuum mechanics provides such a framework!

Continuum mechanics

Treating the material as a continuum, namely a substance that behaves according to some smooth, average behavior, is often more convenient and useful than focusing on the micro-scales. For example, consider the tectonic deformation in Tibet as seen from the style of earthquake faulting and geodetic measurements of crustal velocities.



World Strain Map



2 ways of describing motion:

Lagrangian vs Eulerian description

If we consider velocities in a fluid moving about, then we are often interested in transport of properties, such as temperature anomalies in convection. There are two main ways to describe how a continuous medium (fluid or rock) moves and deforms:

Lagrangian description: we follow individual material particles as they move.

Eulerian description: we stay at fixed points in space and watch material flow past.

Both viewpoints describe the same physics, but they answer slightly different questions.

Lagrangian reference frame

In this frame, we take the point of view of going along for the ride on a fluid parcel that moves through a fixed reference frame, the markers on the sides of the river which pass by us while we sit on a boat.

In the Lagrangian view, we “tag” each material particle by its initial position X at time t = 0. We then follow that particle as it moves. Intuitively: we paint dots on a rock and track where each dot goes and how distances between dots change. This is very natural for describing finite strain and the history of deformation of a given piece of rock.

Eulerian reference frame

In the Eulerian view, we fix a coordinate system in space and do not move with the material. We describe fields as functions of position and time, e.g. a velocity field.

At each fixed point in space, we ask: “what is the velocity / temperature / stress here, now?” Intuitively: we put a measuring station at a fixed location and watch different pieces of rock or fluid flow past it. **This is the standard viewpoint for PDEs and numerical models, where fields are defined on a fixed grid.**

Stress and Strain

Stress

↑

force per unit area

$$\sigma = \frac{F}{A}$$

Unit: N/m^2

Strain

↑

Change in length

$$\varepsilon = \frac{x}{L}$$

x ← $x, \Delta x$

L ← l_0, L

Unit: m/m

Stress

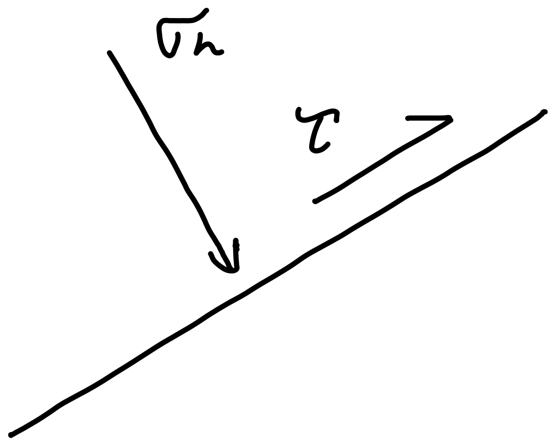
Stress is force acting across an internal surface inside a material. It is not something applied from outside — it exists within the material (for example, rocks).

The force across the plane has two components:

Normal stress (σ_n): pushes or pulls perpendicular to the plane

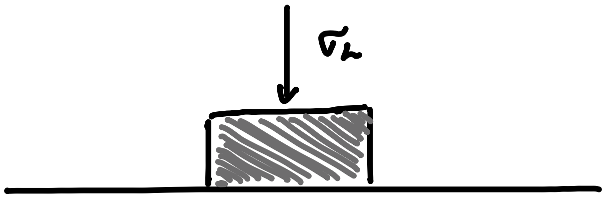
Shear stress (τ): acts parallel to the plane, trying to slide one side past the other

Stress always depends on the orientation of the plane, so a single number cannot describe it. This directional dependence is the reason we will need a **tensor**, but not yet, first understand the physical idea.

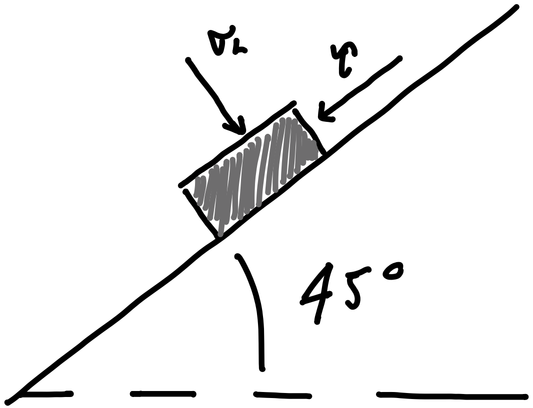


Stress

Why a single number is not enough – stress depends on direction



If you apply the same force to a block (or just by gravity), stresses on different planes are not the same. A horizontal plane might feel only normal stress, a 45° plane might feel mostly shear stress. To fully describe the stress state at a point, we must know:



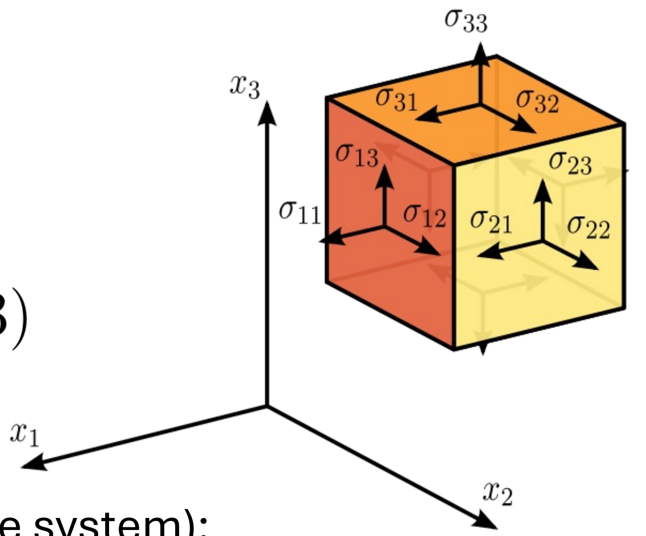
When stress is acting on three perpendicular faces in three perpendicular directions (3D), this requires **nine components.**

These nine components form the **stress tensor**, a compact way to track how force acts on any plane.

Stress

Second-rank tensor

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \qquad (i, j = 1, 2, 3)$$



Invariants (quantities independent of the coordinate system):

- First invariant (trace): $I_1 = \mathrm{tr}(\sigma) = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{kk}$
- Second invariant (magnitude, for a deviatoric tensor):

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\,\delta_{ij}, \qquad \|s\| = \sqrt{s_{ij}s_{ij}} = \sqrt{s_{11}^2 + s_{22}^2 + s_{33}^2 + s_{12}^2 + s_{13}^2 + s_{21}^2 + s_{23}^2 + s_{31}^2 + s_{32}^2}$$

Stress

In continuum mechanics compressional (extensional) stress are negative (positive). Pressure is positive under compression. Stress is measured in Pa = N/m^2. The stress tensor contains the components of the tractions acting on the element surfaces. The first index indicate the direction of stress, the second the normal to the stressed surface

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

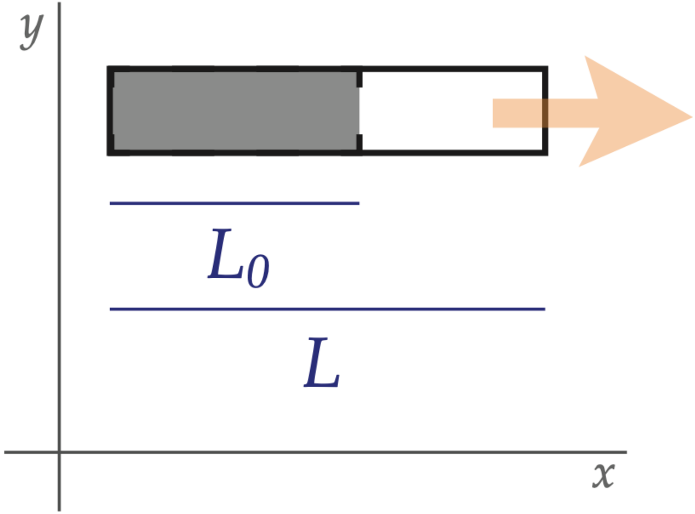
Pressure is equal to the mean normal stress:

$$2D: \quad P = -\frac{tr(\sigma)}{2} = -\frac{\sigma_{kk}}{2} = -\frac{\sigma_{xx} + \sigma_{yy}}{2}$$

$$3D: \quad P = -\frac{tr(\sigma)}{3} = -\frac{\sigma_{kk}}{3} = -\frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}$$

Strain

Strain describes the change in relative position of material points that make up a body. The Illustration shows **elongation**, or **normal strain**.

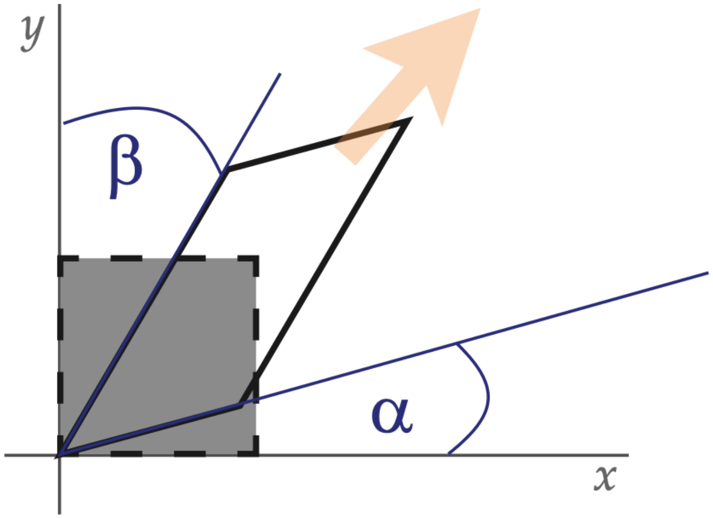


$$\varepsilon_n = \frac{L - L_0}{L_0} = \frac{\Delta L}{L_0}$$

Strain

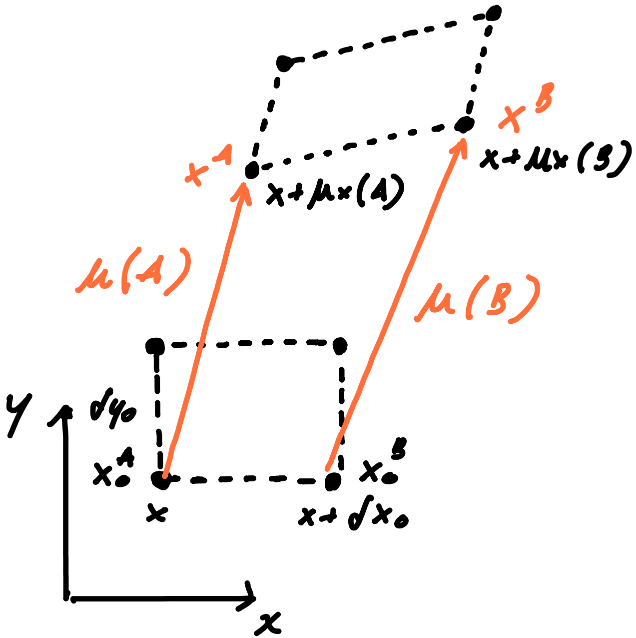
Illustration of **shear strain**. The original square, fixed at the lower left corner, is pulled diagonally, leading to shearing type displacements ux which increase with y, tilting the left, originally vertical, edge of the object at an angle β, as well as uy displacements which increase with x, leading to the bottom, originally horizontal, edge being inclined at angle α.

$$\gamma = \alpha + \beta$$



Strain – derivation of the infinitesimal strain

Derivation of the infinitesimal strain



$$x^A = x_0^A + \mu(A)$$
$$x^B = x_0^B + \mu(B)$$

$$dx = \mu_x(B) - \mu_x(A) + dx_0$$

↳ Normal strain:

$$\varepsilon_n = \frac{\mu_x(B) - \mu_x(A)}{dx_0}$$

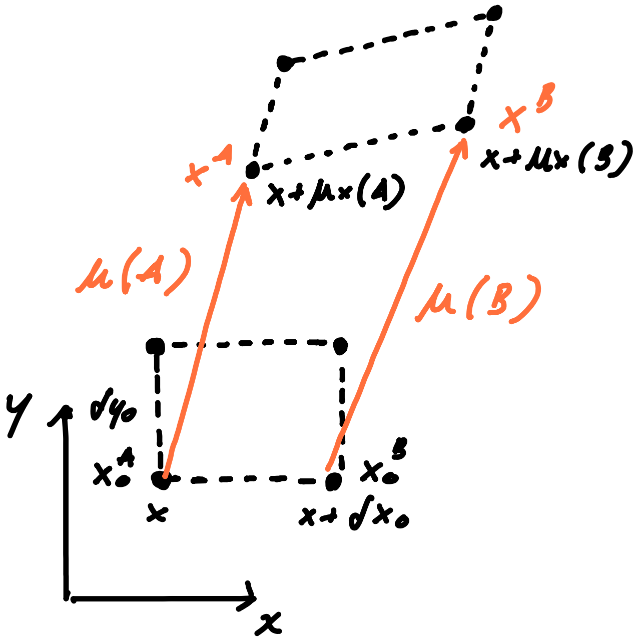
Assuming small $dx_0 = B - A$ (infinitesimal)

$$\mu_x(B) = \mu_x(A) + \frac{d\mu_x(A)}{dx} dx_0$$

$$\varepsilon_n \approx \frac{\mu_x(A) + \frac{d\mu_x(A)}{dx} dx_0}{dx_0} = \boxed{\frac{d\mu_x}{dx}}$$

Strain – derivation of the infinitesimal strain

Derivation of the infinitesimal strain

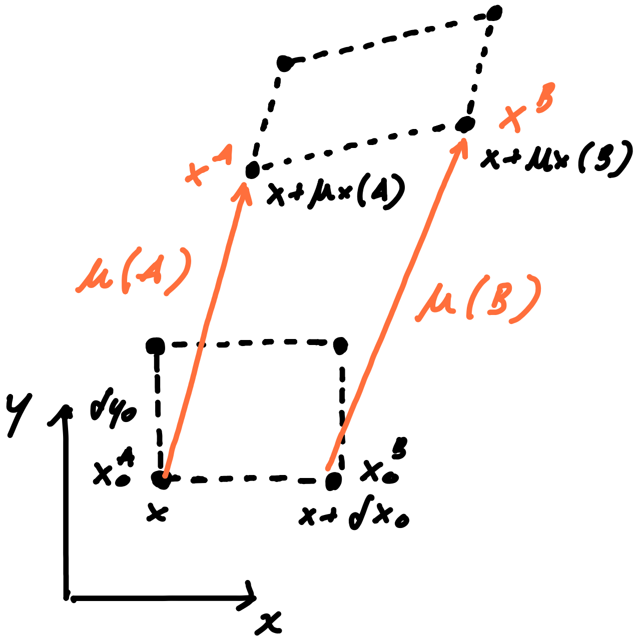


We can use these equations to define the **normal strain in x-direction**, assuming **infinitesimally small** δx0 and small displacements, and in analogy in y, and define:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \quad \text{and} \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

Strain – derivation of the infinitesimal strain

Derivation of the infinitesimal strain



How about the **shear angles** α, β, and γ?

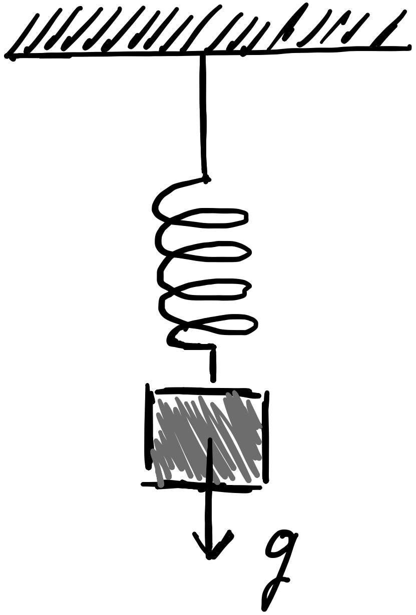
$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$\varepsilon_{xy} = \frac{\gamma_{xy}}{2} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

Elasticity

Hooke's Law: From stress–strain to rock elasticity

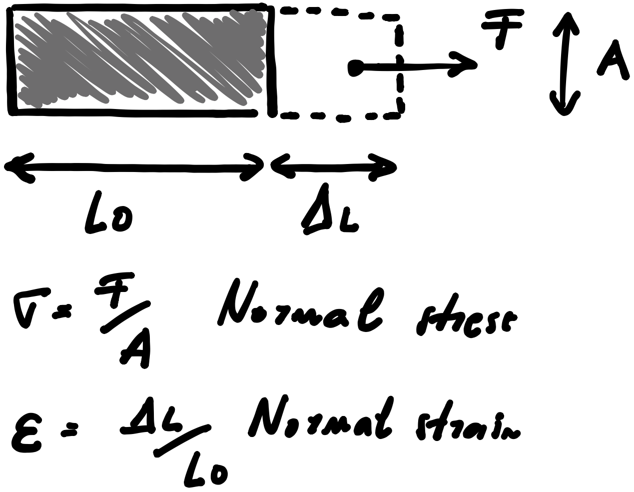
- Hooke's law describes how a material deforms elastically under applied stress.
- In the elastic regime, strain is proportional to stress.
- This relation is the starting point for understanding elastic waves, flexure, and many geophysical processes.



Elasticity

Stress and strain in the elastic regime

Consider a bar with cross-section area A and length L0. Apply a force F along its axis. The bar extends by ΔL.



Elasticity

Hooke's Law in 1D

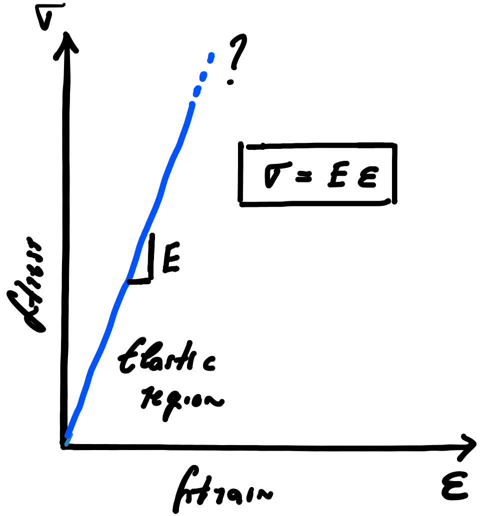
In the elastic regime, many solids obey a linear relationship between stress and strain. ***E*** **is Young's modulus** (stiffness).

- Large E: material is stiff, small strain for a given stress.
- Small E: material is compliant, large strain for a given stress.

$$E = \frac{Pa}{\text{dimensionless}} = Pa$$

Soft rubber $\approx 10^6$ Pa

Steel / Rocks $\approx 10^{10} - 10^{11}$ Pa

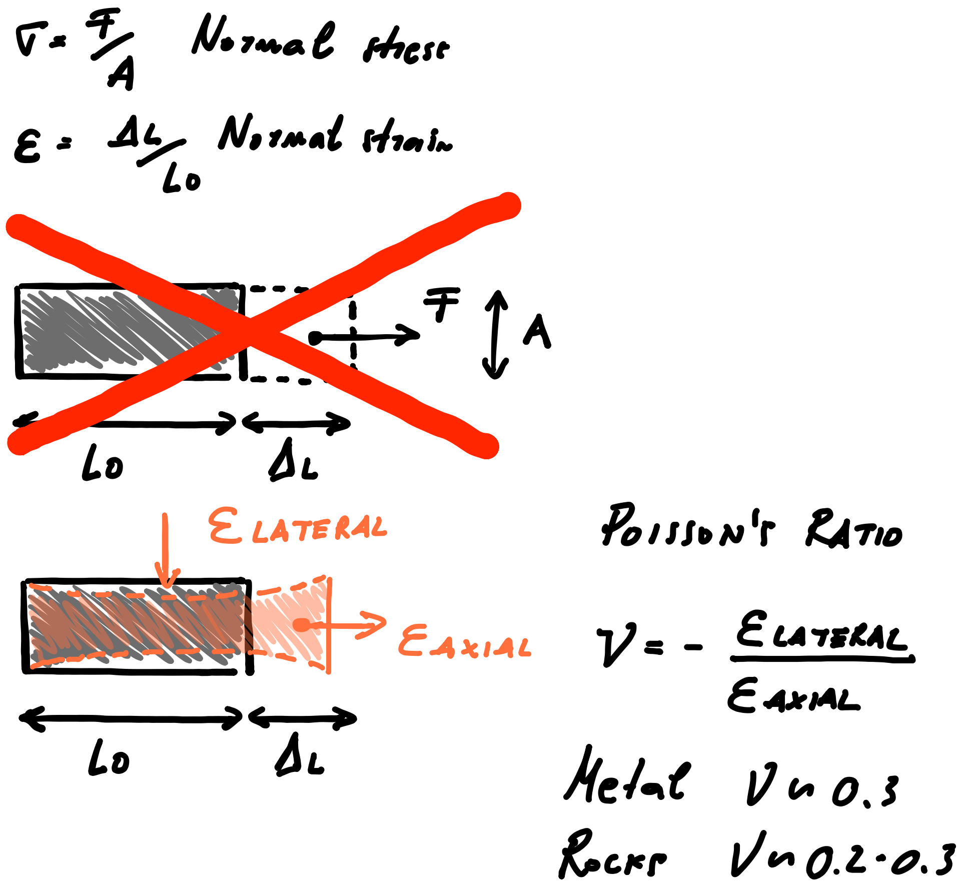


Physically: E measures how much stress is needed to produce a given strain.

Elasticity

Poisson's Ratio:
3D Effects of 1D Loading

Under uniaxial tension, a bar extends in one direction and contracts in the perpendicular directions.



Elasticity

Relating elastic parameters

Elastic constants: E, ν, λ, μ

For isotropic material:

$$E = 2\mu(1+\nu)$$

λ Lamé constants

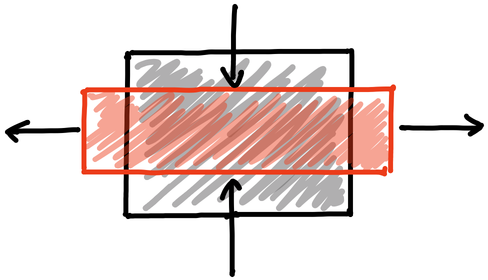
$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

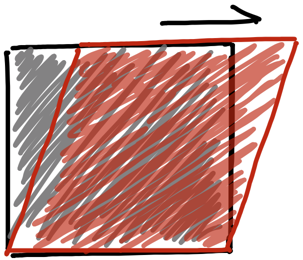
Hooke's law and shear: shear modulus

$$\tau = \mu \varepsilon$$

τ: shear stress; μ: shear modulus; ε: shear strain

Pure shear





Simple shear

*Shear modulus, also known as the **modulus of rigidity**, is a measure of a solid material's resistance to deformation when a force is applied parallel to its surface. It is defined as the ratio of shear stress (force per unit area) to shear strain (the amount of deformation).*

Strain of tectonic plates

The Pacific plate is moving at ~44 mm/yr with respect to North America. Assume that deformation is accommodated in a 150 km wide zone. What is the strain-rate in the shear zone? State your assumptions

Elasticity

What is the difference between young's modulus and shear modulus?

Short version: both measure stiffness, but in different “directions”.

Young’s modulus: stiffness in extension/compression (normal loading).

Shear modulus: stiffness in shear (sideways distortions).

Elasticity

Why Elasticity matters in Geophysics

- Elasticity gives us the first-order rule for how materials respond to small forces: if you don’t understand elastic stress–strain, you can’t make sense of strength, failure, or long-term ductile behavior either. It’s the baseline model everything else deviates from.

- In geophysics, seismic waves are just tiny elastic disturbances propagating through rocks; the elastic moduli directly control P- and S-wave speeds, so seismology is “applied Hooke’s law.”

- Elastic deformation controls how the lithosphere bends under loads (ice sheets, volcanoes, mountain belts), how stress builds up on faults before earthquakes, and how the crust rebounds after events like deglaciation.

- Elasticity is also what lets us invert observations for structure: by measuring wave speeds, deformations, and flexure, we can infer the elastic properties (and thus composition, temperature, and mechanical state) of rocks deep inside the Earth, where we will never drill.

Deviatoric stress

Stress can be divided into a deviatoric and an isotropic components. The deviatoric components produce flow, the isotropic components (i.e., pressure) compaction or dilation.

$$\sigma_{ij} = \sigma'_{ij} + \delta_{ij}\frac{\sigma_{kk}}{3} = \sigma'_{ij} - P\delta_{ij} \quad where \quad \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$3D: \sigma'_{ij} = \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} & \sigma'_{xz} \\ \sigma'_{yx} & \sigma'_{yy} & \sigma'_{yz} \\ \sigma'_{zx} & \sigma'_{zy} & \sigma'_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} + P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} + P & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} + P \end{bmatrix}$$

$$Normal\ deviatoric\ stress: \sigma'_{ii} = \sigma_{ii} + P;$$

$$Shear\ deviatoric\ stress: \sigma'_{ij} = \sigma'_{ji} = \sigma_{ij} = \sigma_{ji}$$

$$Please,\ show\ that\ tr\left(\sigma'_{ij}\right) = 0$$

$$Second\ invariant\ of\ deviatoric\ stress\ tensor:$$

$$\sigma'_{II} = \sqrt{\tfrac{1}{2}\sigma'^2_{ij}} = \sqrt{\tfrac{1}{2}\left(\sigma'^2_{xx} + \sigma'^2_{yy} + \sigma'^2_{zz}\right) + \sigma'^2_{xy} + \sigma'^2_{xz} + \sigma'^2_{yz}}$$

Deviatoric stress

Stress can be divided into a deviatoric and an isotropic components. The deviatoric components produce flow, the isotropic components (i.e., pressure) compaction or dilation.

$$2D:\ \sigma'_{ij} = \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{yx} & \sigma'_{yy} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} + P & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} + P \end{bmatrix}$$

$$Please,\ show\ that\ in\ 2D \Rightarrow \sigma'_{xx} = -\sigma'_{yy}$$

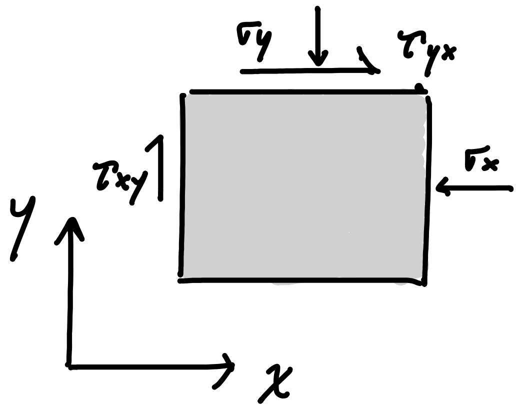
$$Second\ \ invariant\ of\ deviatoric\ stress\ tensor:$$

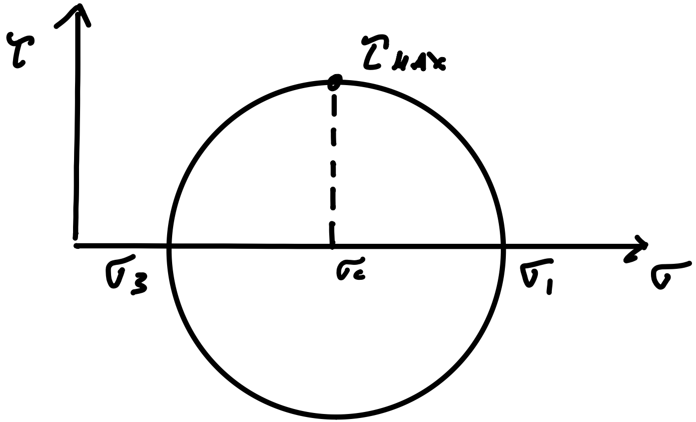
$$\sigma'_{II} = \sqrt{\tfrac{1}{2}\sigma'^{2}_{ij}} = \sqrt{\sigma'^{2}_{xx} + \sigma'^{2}_{xy}}$$

Mohr circle

Up to now, we have described stress using tensors and invariants. But a rock element “feels’’ stress on all possible planes, not only the coordinate axes. The stress tensor contains this information, but it is not obvious how: **normal stress and shear stress vary as a function of plane orientation.**

A transformation of coordinates can compute these values, but Mohr’s circle lets us see the transformation immediately.





Mohr circle

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix}$$

$\sigma_n(\theta)$ and $\tau(\theta)$

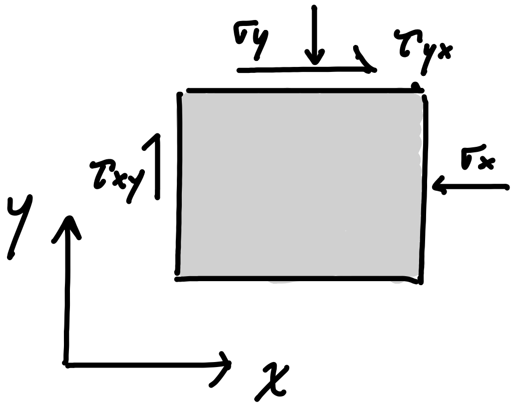
The PRINCIPAL STRESSES occur where

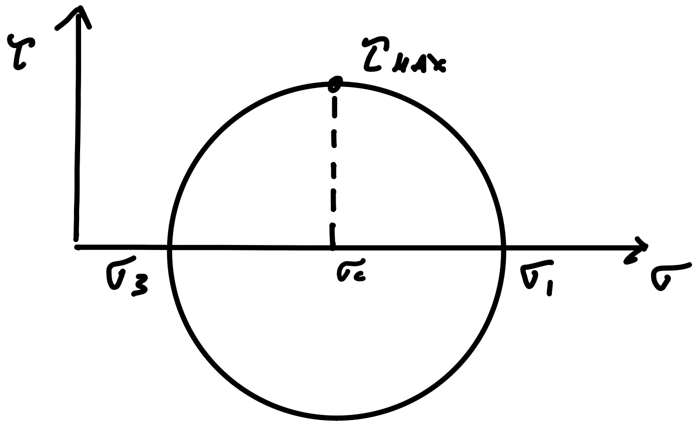
$\tau(\theta) = 0$

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

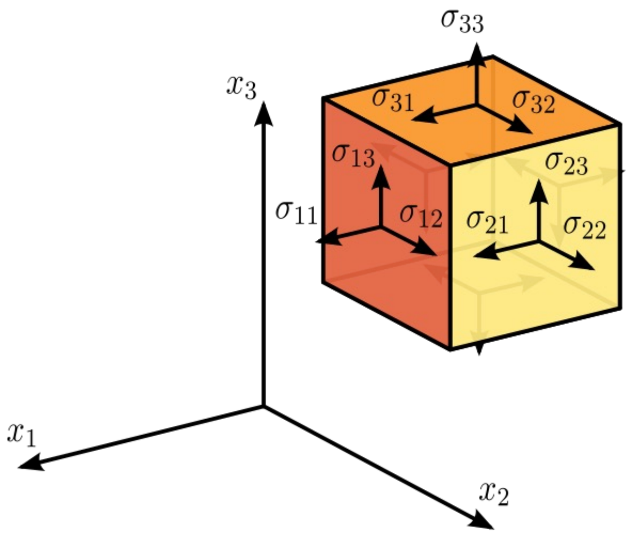
if you know σ_1 and σ_2, you know the entire stress state in 2D.

$$\sigma_c = \frac{\sigma_{xx} + \sigma_{yy}}{2}$$





Recap



This cube represents a tiny piece of rock. We are looking at the forces acting on each face of the cube.

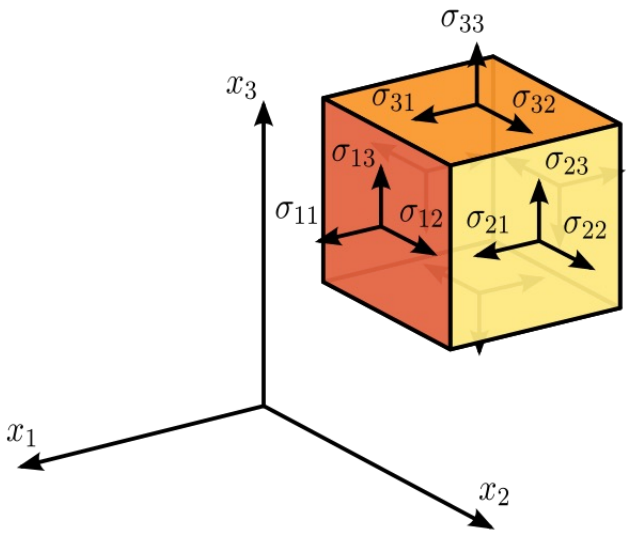
Each face has:

1) A normal direction (perpendicular to the face)

2) forces that can act along different directions on that face.

σ_{ij} means:

i = direction of the force
j = direction of the normal to the face



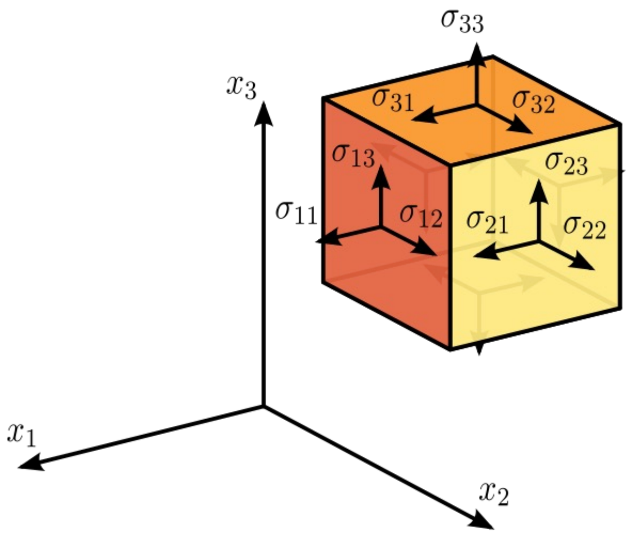
Example: why σ_{31} is called σ_{31}

Point to the cube.

The face labelled 1 is the face whose normal points along x_1.

On that face, there is a force pointing in direction x_3

So:
face normal → 1
force direction → 3



Why the diagonal terms look simpler (σ_{11}, σ_{22}, σ_{33})

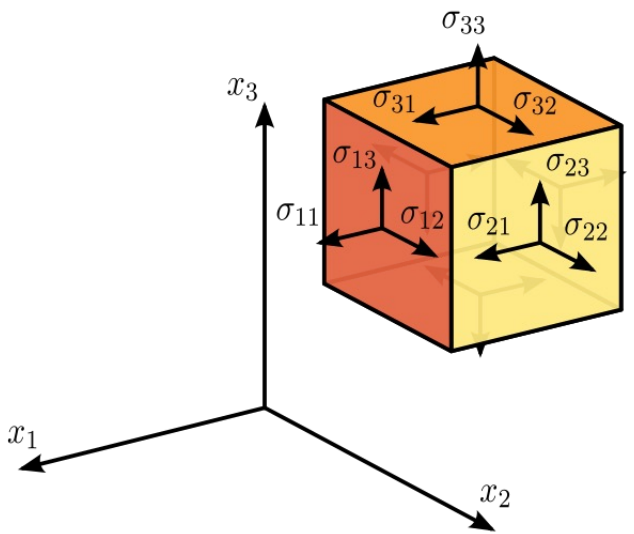
Now connect to intuition:

- σ_{11}: force in x_1 on face normal to x_1
- σ_{22}: force in x_2 on face normal to x_2
- σ_{33}: force in x_3 on face normal to x_3

Translating $\sigma_{12} \leftrightarrow \sigma_{xy}$ (left cube → right matrix)

The cube uses numbers (1, 2, 3). The matrix uses letters (x, y, z).
They describe the same directions.
Mapping:

1 ↔ x
2 ↔ y
3 ↔ z



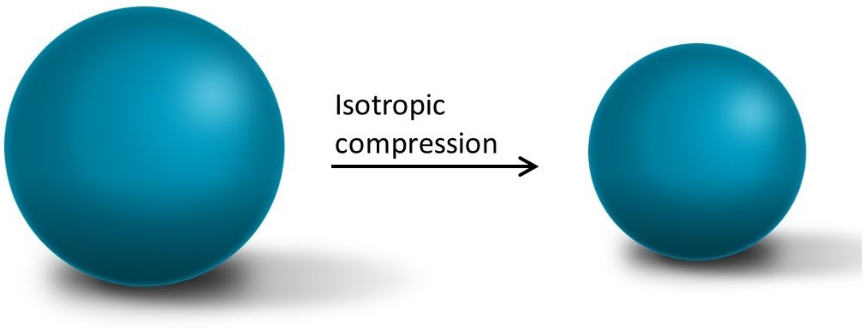
???

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Every stress component answers the question:
Which face am I on, and which way am I pushing?

Isotropic stress (pressure)

Now imagine squeezing a rock equally from all directions: No preferred direction, no shear, no shape change, only volume change.



This part of the stress is called isotropic stress, or **pressure.**

Isotropic stress (pressure)

$$P = -\frac{1}{3}\sigma_{kk} \quad \text{(mean normal stress)}$$

$$\sigma_{kk} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \quad \text{(trace)}$$

The trace it is the sum of the **normal stresses**
dividing by 3 gives the average

Key physical meaning:

Pressure is the part of stress that is the same in all directions.

Isotropic stress (pressure)

$$P = -\frac{1}{3}\sigma_{kk} \quad \text{(mean normal stress)}$$

$$\sigma_{kk} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \quad \text{(trace)}$$

$$3D: \sigma'_{ij} = \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} & \sigma'_{xz} \\ \sigma'_{yx} & \sigma'_{yy} & \sigma'_{yz} \\ \sigma'_{zx} & \sigma'_{zy} & \sigma'_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} + P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} + P & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} + P \end{bmatrix}$$

Why pressure acts only on normal components? This is crucial and often misunderstood. Pressure is directionless. It pushes perpendicular to surfaces. It does not create tangential forces. Therefore, it cannot produce shear.

Total / deviatoric / Effective stress: definition

Total stress = pressure (isotropic) + deviatoric stress (distortional). Total stress is the force per unit area acting inside a rock, including both normal stresses (pressure-like) and shear stresses (all components).

Pressure is the isotropic part of the total stress: the same compressive stress acting equally in all directions.

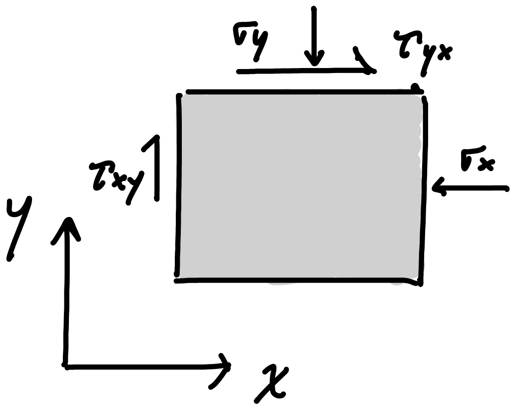
Deviatoric stress is the part of the total stress that remains after removing pressure. Deviatoric stress measures how much stress differs from pure pressure.

Effective stress is the part of the total stress carried by the solid rock framework when fluids are present. Effective stress is defined as the difference between total stress and pore fluid pressure.

Mohr circle

So far, we have described stress using components defined along the x and y axes. **But a rock does not know what *x* and *y* are.**

But… what is the stress acting on a plane that is not aligned with x or y?



Mohr circle

So far, we have described stress using components defined along the x and y axes. **But a rock does not know what *x* and *y* are.**

But… what is the stress acting on a plane that is not aligned with x or y?

Inside the rock, there are infinitely many planes at different angles. Pick one plane, tilted at some angle. On that plane, two things act:
- a normal stress (pushing into or pulling out of the plane)
- a shear stress (trying to make the two sides slide)

The question is: **If I choose a plane at some angle, what are the values of normal stress and shear stress on that plane?**

Mohr circle